Local approximation of meshes by quadrics

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Abstract

This paper deals with local interpolation and approximation of meshes by quadrics. Our goal is to estimate the shapes on meshes, which are closely related to discrete curvatures. Our study aims at analyzing a vertex shape through the properties of a local fitting quadric. One of the most important difficulties in the discrete case is due to artifacts requiring a more detailed classification of vertices than in the continuous case. Our approach proposes a solution and permit a better processing of geometrical data.

Keywords: quadrics, interpolation, approximation, discrete curvatures

1 Introduction

Clouds of points, originating from physical measures or data processing, must be analyzed prior to a triangulation, in order to determine points yielding difficulties during further processings. Local approximation by a quadric is a simple way to obtain indications on geometrical properties in the vertex neighborhood. This approach provides interesting results but requires additional work: analysis of the numerical difficulties which can sometimes be encountered, determination of a good number of neighbors defining the associated system and choice of the best constraints to solve it (see section 3). But the main problem is that the method exhibits the characteristics of a C^2 surface and is thus completely inefficient to detect any of the artifacts encountered in the discrete case, as described in section 3. In order to apply this approach with robustness, an *a priori* local analysis of the geometry of the vertex and its neighbors (number of neighbors, spatial distribution, *etc.*) must be achieved. The techniques developed use both numerical criteria (analysis of singular values and residue) and geometrical ones.

The paper is organized as follows: in section 2, we present a short review of works on discrete curvatures. In section 3, we introduce the principle of our method. Section 4 emphasizes the interesting results we obtained during numerous tests. We conclude in section 5.

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2 Related works

The shape of an object is usually studied in any point with an analysis of its curvatures. All the fundamental results of differential geometry can be found in [DoCarmo 1976]. Analyzing polyedral surfaces implies the definition of second order discrete estimators called discrete curvatures. Mainly two approaches have already been proposed.

The first one is based on the "*angular defect*" computation (see for example [Dyn et al. 2001; Borrelli et al. 2003; Alboul and VanDamme 1997; Alboul and VanDamme 1995; Meyer et al. 2002]). The idea is to propose an analogy of the well-known Gauss definition (see [Bac et al. 2005] for a more detailed presentation). Consider a triangulation of a vertex and its first neighbors. The normalized normal vector to each triangle are called N_i. These vectors can be placed on a sphere and connected by great circle segments (see figure 1) defining the discrete Gaussian indicatrix. The Gaussian discrete curvature is defined by a ratio N/D, where D is a quantity related to the area of the previous triangles and N the area of the spherical polyhedron on the discrete indicatrix. This area is called the angular defect and is defined by $2\pi - \sum \alpha_i$, where the α_i are the angles between two consecutive edges connected to the vertex (figure 1).

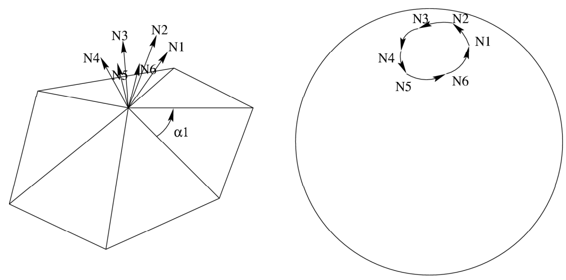

Figure 1: *Angular defect of a point and its Gaussian indicatrix.*

The angular defect can be interpreted as the angle that remains when cutting one edge in order to unfold the faces. $2\pi - \sum \alpha_i = 0$ corresponds to a parabolic or flat point which can be directly unfolded without any cut. $2\pi - \sum \alpha_i > 0$ corresponds to a convex point and when $2\pi - \sum \alpha_i < 0$, there is an overlap of faces in the plane corresponding to a saddle point.

But this approach has limitations due to "artifacts" inexisting in the continuous case. They have been first pointed out in [Alboul and VanDamme 1995]. Alboul and Van Damme classified vertices in 3 categories according to the spherical indicatrix (see figure 2) which are: "Convex vertices" when the indicatrix turns counterclockwise (positive loop), "saddle vertices" when the indicatrix turns clockwise (negative loop), and "mixed" when the indicatrix changes its orientation (there exists at least a self-intersection). We prefer to use the term '*'fan points*" in relation with the shape of such vertices.

A more complete analysis emphasizes a more complex situation [Bac et al. 2005]. A convex vertex and its symmetrical concave one have the same spherical indicatrix (there are differentiated in the continuous case by the values of the principal normal curvatures).

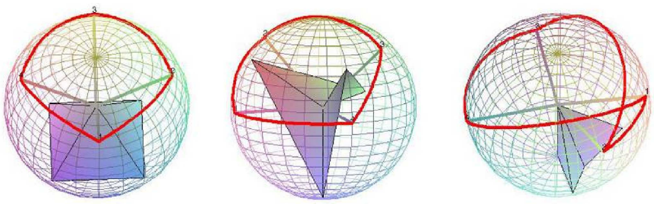

Figure 2: *Convex, saddle and mixed vertices.*

For two contiguous vertices in the discretization, one can be convex, the other concave without intermediate case (one is a fan point). There exist also at least six cases: convex, concave, saddle, fan convex, fan concave and fan saddle. They correspond to completely different situations, one of the most important being the existence of a supporting plane or not. Finally, it is possible to create a fan convex with a negative angular defect as shown in figure 3 (left), while increasing the number of vertices moves the angular defect toward $-\infty$, figure 3 (right).

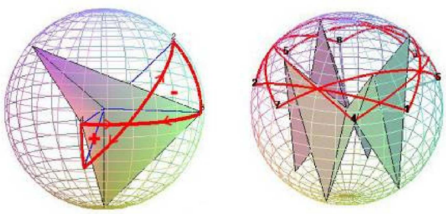

Figure 3: *Fans convex with negative angular defect.*

In addition to the drawbacks previously pointed out, the results controlling the convergence toward the Gaussian continuous curvature when the density of points increases are poor ([Borrelli et al. 2003]). A triangulation must also already be constructed, and the results are highly dependent on this triangulation.

This explains why, even if powerful applications are based on the angular defect calculation, another approach is developed. A local fitting with a low degree polynomial surface, mainly a "*quadric*" is computed (see [Berger and Gostiaux 1988; Buxton and Douros 2002; Cipolla and Giblin 2000; Lane 1940; Daniel et al. 2007]). Approximation with quadrics has been used by many authors with different goals, one of the first main contributions being ([Sander and Zucker 1990]). A global reconstruction by adjusting local estimators for quadrics (principal curvatures and principal directions) was proposed there. Recently, improvements have been made by developing tests to choose between global approximation by a general quadric, local approximation in a local coordinate system or particular treatment for edges and corners ([Ohtake et al. 2003]); the results are weighted to find a complete model in an implicit form. In this model, extraordinary points have been removed to receive a smooth object. These approaches often have a common point: they use normals at each vertex, either by estimation or because they are given as data.

We on the other hand want to use raw points, without characterization of normals or triangulation. This choice is motivated by the fact that computation of normals, coming from some technique of geometric approximation (weighted mean, Voronoï cells, estimation of tangent plane) is rarely very reliable and is not necessarily a good starting point for second order estimations. Moreover, the reality of digitized data sets is the existence of noise, which raises problems in the normal vector estimation process and which often leads to artifacts we try to detect.

Once a local quadrics is obtained, the local differential characteristics of this surface can be considered as relevant local discrete estimators. The principal curvatures k_1, k_2, the principal directions, the Gaussian and mean curvatures can be defined. They evidently must be carefully interpreted, taking into account the switch between the discrete and continuous spaces. As a consequence, the method exhibits the characteristics of a C^2 surface and is thus completely inefficient to detect any of the artifacts described above, without any additional analysis.

The following sections will present our approach to fit with a quadric and how we can detect the different situations. Some works deal with "cubics" ([Cazals and Pouget 2005]), which allows a wider system of classification. We will focus on quadric fitting which is faster in terms of calculations and easier to classify.

3 Fitting with a quadric

A general quadric is represented by a homogeneous equation with 10 coefficients. As explained in the following part, one of these coefficients will be conditionally fixed. Linear equations are obtained by writing conditions for a quadric to contain 9 (or more than 9) given points. Thus, the quadric is determined by 9 equations for interpolation, or more than 9 equations for approximation.

The general equation of a quadric in homogeneous coordinates is $V^tQV = 0$, Q being a symmetric 4×4 matrix. The developped form in cartesian coordinates is $F(x, y, z) = ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + 2gx + 2hy + 2iz + j = 0$ and the points that we propose to be on a quadric must satisfy:

$$F(x_i, y_i, z_i) = 0, \; i = 1 \text{ to } n \tag{1}$$

We then obtain the linear system: $Ax = B$ represented in matrix form by:

$$\begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & 2x_1y_1 & 2x_1z_1 & \dots & 1 \\ x_2^2 & y_2^2 & z_2^2 & 2x_2y_2 & 2x_2z_2 & \dots & 1 \\ \vdots & & & & & & \vdots \\ x_n^2 & y_n^2 & z_n^2 & 2x_ny_n & 2x_nz_n & \dots & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ \vdots \\ j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

To solve the system of equations we use a Singular Value Decomposition (SVD in the following) ([Ciarlet 1998]) which have two advantages: robustness in pathological or degenerated cases, easiness for fine analysis of conditioning. We can study then some related issues like the condition number, the residue, singular values, singular vectors.

In order to treat the problem of homogeneity, Buxton and Douros in [Buxton and Douros 2002] proposed the constraint ($a^2 + b^2 + c^2 + 2d^2 + 2e^2 + 2f^2 + 2g^2 + 2h^2 + 2i^2 + j^2 = 1$), but it implies more calculations. In order to keep the linear system approach, we chose to set arbitrarily a coefficient, and in particular to set the coefficient j of the quadric equation to -1 (we chose j which is a factor independent of the coordinates).

To experimentally check, we tested the choice of value j, in all the cases and for all the quadrics. With respect to the condition number value there was no change. On the other hand, the coefficients of the quadric were multiplied by $-j$ and the residue value by $-j^2$, which was predictable. The only problem with this choice is when the quadric passes through the origin. This issue will be studied in paragraph 4.2.

We need at least 9 points to get a system of equations giving solutions that does not depend on a parameter. In fact, we can take a neighborhood of more than 9 points in order to get a better uniform distribution around the point and to have greater smoothing if desired. However, we must be aware of the risks of taking an important number of neighbors. On the one hand, more points allow better smoothing of a possible noise. On the other hand, more points involve as a matter of fact points which are far away from our vertex and thus the risk of losing local information which is nevertheless the characteristic of the analysis of shapes (the differential geometry works in the neighborhood of a vertex). The neighbors are determined through a distance computation from the given vertex.

From a theoretical point of view all condition numbers lower than infinity correspond to a matrix of maximum rank. In practice, considering the computations, the condition number increase indicates quickly a problem of rank numerical degeneracy. Only the order of magnitude of the condition number is relevant (see [Daubisse 1984]). In practice, condition numbers of 100 and 150 are considered to be equivalent, while values 100 and 1000 are different (10^2 et 10^3). As all manipulated values are approximated (accuracy of the digitized or computed data, approximation errors, calculations rounding, ...), studying the condition number is necessarily very important.

We programmed using double precision floating arithmetic. 10^{-15} is the order of magnitude of the rounding error. As a consequence, a condition number greater or equal to 10^{15} can be considered as infinity: there is at least one vanishing singular value compared to the largest one.

To analyze the shapes of determined quadrics, we use the eigenvalues of the 3×3 matrix of degree 2 terms, which, together with constant term, enable to have a reduced form of the equation. Eigenvalues of same sign determine an ellipsoid and opposite signs determine an hyperboloid, either with one sheet or two sheets. A null eigenvalue indicates a paraboloid in presence of first degree terms otherwise a cylinder. Finally, a degenerate case corresponds to the situation where the 4×4 matrix Q is singular. (see [Berger and Gostiaux 1988] for typology of quadrics).

4 Results

Sub-sections 4.1 to 4.4 exhibit the results we obtained to detect the different configuration of point shape encountered on digitized object.

4.1 General Tests

We classify the quadrics according to standard typology:

1. The so-called genuine quadric surfaces that correspond to the great family of quadric surfaces, like: hyperboloid with two Sheets (H2), hyperboloid with one Sheet (H1), ellipsoid (E), hyperbolic paraboloid (HP) and elliptic paraboloid (EP).
2. Individual cases of known surfaces that are not classified as degenerated surfaces but do not pertain to the previous category, like: hyperbolic cylinder, parabolic cylinder, elliptic cylinder and cone.
3. And finally all degenerated quadric surfaces, like two intersecting planes, two parallel planes, a plane, a line, a point and the empty set.

The first sequence of tests is, given a vertex on a quadric, discretize it to obtain the vertex neighbors, then fit this set of points as described in section 3 and analyze the result. Different types of quadric surfaces have been considered and the following characteristics have been considered: the condition number, the residual, the singular values, the singular vectors, the eigenvectors, the point distribution around the vertex. The various examples of quadric surfaces lead to the same results.

A typical example is proposed with an ellipsoid, $\frac{x^2}{2} + \frac{y^2}{2} + z^2 = 1$, with irregular distributions of distance and angle around the vertex. The condition number is compatible with correct results. Table 1 collects the coefficients of the ellipsoid. The eigenvalues and the constant j which is equal to 1 are characteristics of an ellipsoid.

The quadric with $j = -1$	
$Cond_2(A) = 1.112517 \times 10^4$	
a	0.5
b	0.5
c	1
d	-1.80438×10^{-15}
e	-1.50276×10^{-14}
f	4.86293×10^{-14}
g	1.46562×10^{-14}
h	-4.54661×10^{-14}
i	-4.95171×10^{-13}
Residual $= 3.9409 \times 10^{-15}$	
λ_1	1
λ_2	0.5
λ_3	0.5
Classification : ellipsoid	

Table 1: *Ellipsoid with irregular distance and angle distributions*

In general, it was noticed that the more regular the points are around the vertex, the worse the condition number. The condition number is closely linked with the regularity of neighbor distribution. In extreme cases, having a regular sampling around the vertex may produce a rank deficiency (the symmetries entail linear combinations of lines). This situation is not only an academic vision since laser digitizing machine can produce regular set of digitized points. Taking a second round of neighbors can solve this problem by increasing the number of points and overcoming the problem of linear combinations. As already mentioned in section 3, increasing the number of neighbors also permits to decrease the noise influence but one must be clearly aware that this approach is only relevant for dense set of points. As a matter of fact, the fitting must absolutely remain a local one.

In order to illustrate the previous problem let us consider the ellipsoid $\frac{x^2}{2} + \frac{y^2}{2} + z^2 = 1$, with 8 points regularly distributed around the vertex. The quadric system provides the results given in table 2.

Adding one point at a time chosen in a second circle of neighbors, the number of small singular values decreases by one for each point added as can be seen in table 3. The condition number is finally correct.

It can occur that increasing the number of neighbors does not improve the condition number. In such a case, the problem is not linked with the point distribution but with other difficulties as it is explained in the following sections.

4.2 Quadric surfaces passing through the origin

A quadric which passes through the origin does not have the free coefficient j in the equation, this coefficient vanishes for this type of surfaces. Fixing $j = -1$ will raise a problem. This situation must be detected in order to be able to fix another coefficient in the equation.

The quadric with $j = -1$	
$Cond_2(A) = 8.055085 \times 10^9$	
a	0.5
b	0.5
c	1
d	-1.50823×10^{-11}
e	-4.78182×10^{-10}
f	-1.10959×10^{-9}
g	4.62764×10^{-10}
h	1.07357×10^{-9}
i	3.29437×10^{-8}
Residual $= 1.04738 \times 10^{-15}$	
λ_1	1
λ_2	0.5
λ_3	0.5
Classification : ellipsoid	

Singular Values
6.28594
2.80249
2.66991
0.505404
0.347779
0.186733
7.80369×10^{-10}
6.01801×10^{-5}
5.24592×10^{-5}

Table 2: *Ellipsoid with 8 points regularly distributed around the vertex*

Adding	1^{st} point	2^{nd} point	3^{rd} point
$Cond_2(A)$	1.2×10^6	1.2×10^6	1.5×10^3
a	0.5	0.5	0.5
b	0.5	0.5	0.5
c	1	1	1
d	-4.3×10^{-14}	1.1×10^{-13}	-1.8×10^{-15}
e	7.3×10^{-13}	3.3×10^{-12}	-5.3×10^{-15}
f	-4.6×10^{-12}	1.6×10^{-12}	-7.3×10^{-15}
g	-6.9×10^{-13}	-3.1×10^{-12}	5.5×10^{-15}
h	4.3×10^{-12}	-1.6×10^{-12}	6.9×10^{-15}
i	-5.8×10^{-12}	-1.9×10^{-11}	3.2×10^{-14}
Singular Values	6.5373	6.86435	7.13032
	3.61568	4.04667	4.07827
	2.74951	3.06364	3.80877
	0.790143	0.977407	0.989041
	0.500267	0.790076	0.871103
	0.268158	0.304277	0.705743
	0.0766955	5.66×10^6	0.00459343
	5.34×10^6	0.0766941	0.0779594
	5.24×10^5	0.0796133	0.104754
Residual	3.46×10^{-15}	2.58×10^{-15}	2.36×10^{-15}

Table 3: *Adding points around the vertex*

The solution is exposed with the hyperbolic paraboloid (HP): $x^2 - y^2 = z$. The results are gathered in table 4.

Computations are achieved in double precision (64 bits). The order of magnitude of the condition number (greater than 10^{15}) clearly indicates a rank deficiency which is confirmed by one singular value numerically equals to 0 compared to the largest one. As explained in the previous section, an attempt to increase the number of neighbors does not modify the results. We also noticed that the solution found through the SVD (one solution is always provided) does not fit well since the residual is not really small. This behavior is not so surprising since the fitting quadric cannot pass through the origin. It is reasonable to try to fit the point set using another equation of quadric. For that purpose, we analyzed the singular vector associated with the vanishing singular value (see table 5).

We noticed, but it is just an observation, that the non-vanishing coordinates exactly correspond to the coefficients of the quadric equation we started from (multiplied by a constant). We can set one of the non-zero coefficients equal to -1 instead of $j = -1$ in the quadric equation and recompute the SVD. We fixed in the previous example coefficient $a = -1$. Table 6 gathers the results. The condition number and the residual are now correct and the right quadric equation is received as a result. This approach is only a heuristic. Another solution could have been to fix sequentially one coefficient

The quadric with $j = -1$	
$Cond_2(A) = 3.268193 \times 10^{17}$	
a	-2.10225
b	-1.33333
c	-0.687117
d	-5.57331×10^{-16}
e	1.54601
f	4.72253×10^{-16}
g	1.89571
h	3.82529×10^{-16}
i	-1.53783
Residual $= 0.078326$	

Singular values
15.5412
2.81943
3.43523
0.441707
0.298576
4.75529×10^{-17}
0.0517809
0.751543
0.0968345

Table 4: *Hyperbolic paraboloid*

Singular Vector
-0.666667
0.666667
7.48479×10^{-16}
8.12592×10^{-16}
-1.48133×10^{-15}
-4.48012×10^{-16}
-5.64899×10^{-16}
-4.86195×10^{-16}
0.333333

Table 5: *Singular vector corresponding to the vanishing singular value*

(a, b, c, etc.) to -1 until a correct result is received. This systematic algorithm is reliable but is not a neat solution and we advice to use the heuristic.

The quadric with $a = -1$	
$Cond_2(A) = 1.046961 \times 10^3$	
b	1
c	1.10593×10^{-15}
d	-7.15555×10^{-16}
e	-1.76655×10^{-15}
f	6.08581×10^{-16}
g	4.13131×10^{-15}
h	-4.27765×10^{-18}
i	0.5
j	-3.05256×10^{-15}
Residual $= 8.49828 \times 10^{-15}$	

Table 6: *Hyperbolic paraboloid with $a = -1$*

In conclusion, a high condition number with one vanishing singular value and a residual numerically not equal to zero, one can figure out that the quadric passes through the origin. The singular vector corresponding to the vanishing singular value, suggests which are the non-vanishing coefficients of the quadric. One of them can be fixed instead of j and the SVD can be recomputed.

4.3 The degenerated quadric surfaces

The most interesting degenerated quadrics in our context are the two intersecting planes, the two parallel planes and a plane. As a matter of fact, many manufactured objects include pieces of planes in their shape, and depending on the digitizing process the previous three cases of degenerated quadrics can be encountered.

Fitting a set of points pertaining to 2 intersecting planes or 2 parallel planes is computed without problem and such degenerating quadrics can not be detected by studying the condition number or the singular values. But the classification of quadrics with respect to their eigenvalues (see section 3) provides the required information.

There is obviously a problem if the number of points in one plane is less that 3, since at least three points are necessary to define a plane. In the case of 2 planes (intersecting or parallel), we looked out to the distribution of points on the 2 planes, considering we handle a minimal set of 9 points. In fact 4 points can lay in one plane and 5 on the other, 3 and 6, 2 and 7, 1 and 8. While cases $4-5$ and $3-6$ do not produce vanishing singular values, cases $2-7$ and $1-8$ can obviously not lead to the correct results. One singular value vanishes in case $2-7$, two in case $1-8$ and 3 in case $0-9$. In practice, the SVD finds two planes, one being the plane defined with the higher number of points and the other is one of the infinity of planes which contain the remaining points.

In $\mathbb{R}^3$ the general second degree equation of a plane is the product of 2 plane equations (parallel or intersecting). When three singular values vanish and the distribution of neighbors does not correspond to a singular one as described in section 4.1, a degenerated quadrics is found: all the points pertain to one plane. As a result, the product of two planes is received, one of them being expected, as illustrated is the following example.

Let us consider nine points laying on the plane, $x+y+z=1$. The results are given in table 7

The quadric with $j=-1$	
$Cond_2(A) = 2.353903 \times 10^{17}$	
a	0.111111
b	0.111111
c	0.111111
d	0.111111
e	0.111111
f	0.111111
g	0.444444
h	0.444444
i	0.444444
Rsidu $= 3.17337 \times 10^{-15}$	

Singular Values
8.34377
2.87435
2.33408
0.791703
0.602589
0.238302
2.49502×10^{-16}
7.88942×10^{-17}
3.54465×10^{-17}

Table 7: *One plane*

Normalizing the coefficients of this quadric yields the equation:

$$x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 8x + 8y + 8z - 9 = 0$$

A formal calculus procedure gives the factorization:

$$(x+y+z-1)(x+y+z+9) = 0$$

$$\Longrightarrow x+y+z = 1 \text{ or } x+y+z = 9$$

It is easy to analyze which plane must be kept (by checking which equation vanishes for one data point).

4.4 The fan points

Fan points, which have no equivalent in the continuous case, raise the most important issue in shape analysis of polyhedral surfaces. These points must be detected as soon as possible in the process in order to avoid loosing them during the forthcoming steps. A first easy detection is obtained if the points belong to 2 sheets of one quadric. Experiments prove that this situation is not the most frequent. All the fitted points can be on a quadric, but not inside a unique neighborhood of the vertex. To illustrate the situation, let us consider a very flat ellipsoid. Points on each side of this quadric are very close together but absolutely not in the same neighborhood. This configuration can easily be detected by considering the angles between the normal vectors on the quadric for the different fitted points. Consequently, we call "*x-opposite*" point, a point where its normal vector on the quadric forms an obtuse angle with the normal on the quadric at point x. In other words, they belong to two different neighborhoods.

We will illustrate our approach with two examples, a convex vertex and a saddle vertex. Starting from these vertices, we will move the position of one or more points to obtain a fan convex and a fan saddle respectively. At the vertex, we will calculate the angular defect and its Gaussian indicatrix. We also calculate the dot products between the normal vector on the quadric at the vertex and the normal vector for all the neighbors. It provides information concerning the existence of *vertex-opposite* points on the quadric.

Let us first consider an elliptic paraboloid, $x^2+y^2+z=1$, with 8 neighbors. As illustrated in figure 4, the Gaussian indicatrix has a positive loop, characterizing a convex vertex.

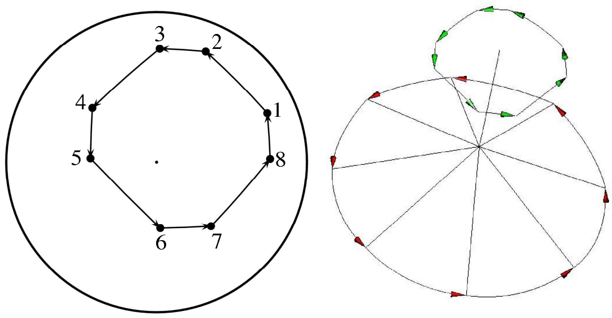

Figure 4: *Gaussian indicatrix of the elliptic paraboloid*

Table 8 groups the information concerning the quadric received from system (1) and the different dot products. The results exactly correspond to a convex vertex: the angular defect is positive and there is no *vertex-opposite* point.

The quadric with $j=-1$	
$Cond_2(A) = 6.098853 \times 10^{02}$	
a	1
b	1
c	1.51851×10^{-14}
d	-6.09898×10^{-17}
e	2.99573×10^{-15}
f	5.86715×10^{-15}
g	-2.86627×10^{-15}
h	-4.61138×10^{-15}
i	0.5
Residual $= 2.92897 \times 10^{-15}$	
Classification: Elliptic paraboloid	

i	$\nabla F\|_{P_0} \cdot \nabla F\|_{P_i}$
1	1.28
2	1.33456
3	1.28
4	1.08
5	0.88
6	0.825442
7	0.88
8	1.08
$2\pi - \sum \alpha_i = 0.577333$	

Table 8: *Convex elliptic paraboloid*

We move point n. 2 to produce a fold between two faces, as shown in figure 5. The Gaussian indicatrix has a positive loop and a negative loop, corresponding to a fan convex.

Table 9 proposes the equivalent results as those proposed in table 8. An ellipsoid is obtained. The condition number is reasonable and the angular defect remains positive. On the other hand, point n. 2 is a *vertex-opposite*. The convex fan is clearly detected.

We will now move two points (points n. 2 and n. 6) to produce 2 folds, as shown in figure 6. The Gaussian indicatrix has a positive loop and 2 negative loops. There exists a supporting plane. The vertex is a fan convex.

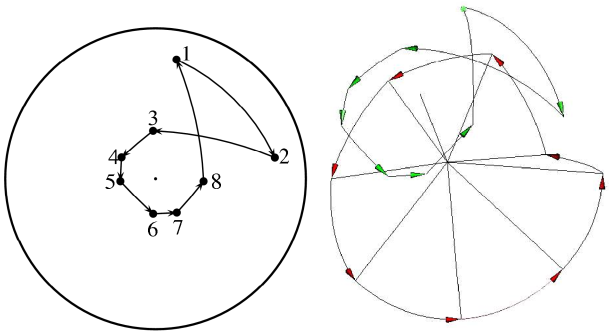

Figure 5: *Gaussian indicatrix of the first fan convex*

The quadric with $j = -1$	
$Cond_2(A) = 4.862589 \times 10^{02}$	
a	1.7175
b	1.7175
c	1.19749
d	-0.161397
e	-0.141541
f	-0.141541
g	0.099979
h	0.099979
i	-0.0845403
Residual = 1.26097×10^{-15}	
Classification: Ellipsoid	

i	$\nabla F\|_{P_0} \cdot \nabla F\|_{P_i}$
1	2.94113
2	-1.88982
3	2.94113
4	3.60752
5	3.80382
6	4.15653
7	3.80382
8	3.60752
$2\pi - \sum \alpha_i = 0.335246$	

Table 9: *First fan convex*

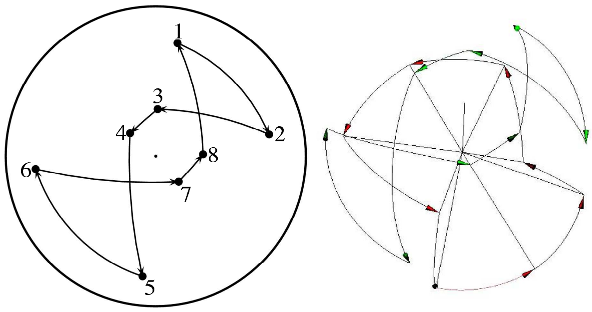

Figure 6: *Gaussian indicatrix of the second fan convex*

The results are presented in table 10. Points 2 and 6 are *vertex-opposite* points. The vertex is a fan vertex. Notice that using the angular defect would have classified the point as saddle.

Let us now consider an hyperbolic paraboloid, $\frac{x^2}{4} - \frac{y^2}{4} + \frac{1}{4} = z$, with 8 neighbors. As illustrated in figure 7, the Gaussian indicatrix has a negative loop characterizing a saddle vertex. Table 11 groups the information concerning the quadric and the dot products. The results exactly correspond to a saddle vertex: the angular defect is negative and there is no *vertex-opposite* point.

We move two points (points n. 3 and n. 7) in order to create folds between faces as illustrated on figure 8. The Gaussian indicatrix has changes in its orientation. The vertex is now a fan saddle. An hyperboloid with two sheets is obtained (see table 12). Four points (points no. 1, 3, 5 and 7) are on one sheet, while the others are on the second sheet. The angular defect is still negative.

Finally, our approach with a fitting quadric allows us to determine fan convex and fan saddle vertices.

The quadric with $j = -1$	
$Cond_2(A) = 1.371724 \times 10^{02}$	
a	3.34677
b	3.34677
c	2.84647
d	-0.0213485
e	0.569294
f	0.569294
g	-0.495997
h	-0.495997
i	-0.931133
Residual = 2.44753×10^{-15}	
Classification: Ellipsoid	

i	$\nabla F\|_{P_0} \cdot \nabla F\|_{P_i}$
1	13.1847
2	-9.24905
3	13.1847
4	11.8929
5	8.29715
6	-15.4699
7	8.29715
8	11.8929
$2\pi - \sum \alpha_i = -0.150051$	

Table 10: *Second fan convex with a negative angular defect*

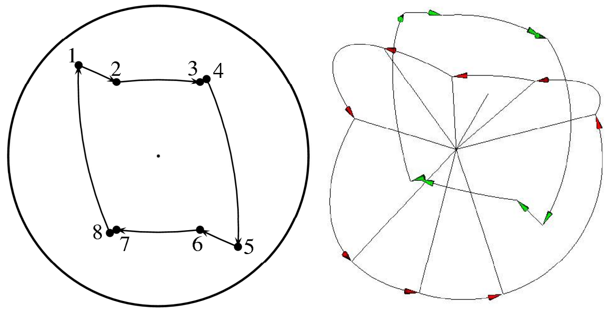

Figure 7: *Gaussian indicatrix of the hyperbolic paraboloid*

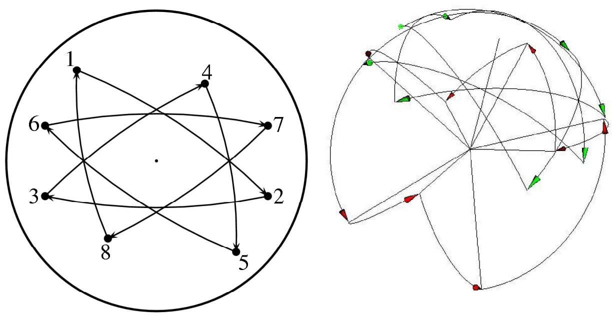

Figure 8: *Gaussian indicatrix of the fan saddle*

5 Conclusion

We propose a local fitting with a quadric, in order to analyze the shape of mesh vertices. Discrete curvatures are deduced from the differential values on the quadric. The classical approach using the angular defect has insufficiencies in pathological configurations of points. Computing a quadric enables us, with an additional geometrical analysis, to receive promising results on the artifacts encountered on vertex shapes.

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The quadric with $j=-1$	
$Cond_2(A) = 2.653029 \times 10^{01}$	
a	-4
b	4
c	7.95479×10^{-16}
d	-1.52706×10^{-16}
e	-3.85248×10^{-16}
f	2.67282×10^{-16}
g	2.39647×10^{-17}
h	1.79713×10^{-16}
i	2
Residual $= 3.68387 \times 10^{-15}$	
Classification: Hyperbolic paraboloid	

i	$\nabla F\|_{P_0} \cdot \nabla F\|_{P_i}$
1	16
2	16
3	16
4	16
5	16
6	16
7	16
8	16
$2\pi - \sum \alpha_i =$ -1.58001	

Table 11: *Saddle hyperbolic paraboloid*

The quadric with $j=-1$	
$Cond_2(A) = 5.971166 \times 10^{01}$	
a	4.56164
b	1.36565
c	-10.5374
d	-1.97894×10^{-16}
e	5.39131×10^{-17}
f	8.14518×10^{-17}
g	-5.40481×10^{-16}
h	3.25683×10^{-16}
i	3.31718
Residual = 7.53971×10^{-15}	
Classification: Hyperboloid with two sheets	

i	$\nabla F\|_{P_0} \cdot \nabla F\|_{P_i}$
1	-15.4035
2	14.8163
3	-8.49609
4	14.8163
5	-15.4035
6	14.8163
7	-8.49609
8	14.8163
$2\pi - \sum \alpha_i =$ -3.27221	

Table 12: *Fan saddle*

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